ON THE k-SYSTEMS OF A SIMPLE POLYTOPE

BY

MICHAEL JOSWIG, VOLKER KAIBEL AND FRIEDERIKE KÖRNER

Technische Universität Berlin, Fakultät II, Institut für Mathematik
MA 6–2, Straße des 17. Juni 136, 10623 Berlin, Germany
e-mail: {joswig,kaibel,koerner}@math.tu-berlin.de

ABSTRACT

A k-system of the graph $\mathcal{G}_P$ of a simple polytope P is a set of induced subgraphs of $\mathcal{G}_P$ that shares certain properties with the set of subgraphs induced by the k-faces of P. This new concept leads to polynomial-size certificates in terms of $\mathcal{G}_P$ for both the set of vertex sets of facets and for abstract objective functions (AOF) in the sense of Kalai. Moreover, it is proved that an acyclic orientation yields an AOF if and only if it induces a unique sink on every 2-face.

1. Introduction

A celebrated theorem of Blind and Mani [2] states that the combinatorial type of any simple polytope P is determined by the isomorphism class of its abstract vertex-edge graph $\mathcal{G}_P$. Kalai [8] gave a short and very elegant proof of this result. The proof is constructive, but the algorithm that can be derived from it has a worst-case running time which is exponential in the size of $\mathcal{G}_P$ (for computational experiments see Achatz and Kleinschmidt [1]). Thus, the complexity status of the problem of reconstructing the combinatorial type of a simple polytope from its graph remains unclear.

Kalai's proof is based on an ingenious characterization of the shellings of the boundary of the dual polytope P^Δ. Each shelling order of the facets of ∂P^Δ corresponds to a linear extension of an acyclic orientation of $\mathcal{G}_P$ which induces a unique sink in each non-empty face. Such a linear ordering of the vertices is called an **abstract objective function**, while the corresponding orientation is

an **AOF-orientation**. Abstract objective functions generalize linear objective functions in general position. The crucial step in Kalai's proof is the characterization of AOF-orientations as those acyclic orientations of $\mathcal{G}_P$ which minimize a certain integer-valued function $\mathcal{H}(\mathcal{O})$. Its minimum value is the total number of non-empty faces of P.

We consider a refinement of the function $\mathcal{H}(\mathcal{O}) = \sum_{k=0}^{d} \mathcal{H}_k(\mathcal{O})$ as a sum of $(d+1)$ functions $\mathcal{H}_0(\mathcal{O}), \dots, \mathcal{H}_d(\mathcal{O})$. This refinement becomes useful in connection with the concept of a k-system that we propose. A k-system of the graph $\mathcal{G}_P$ of a simple d-polytope P is a set of induced subgraphs of $\mathcal{G}_P$ satisfying simple combinatorial conditions (that can be checked in polynomial time) that, in particular, are fulfilled by the set of subgraphs induced by the k-faces. Our main result on k-systems (Theorem 1) is that, on the one hand, a k-system of the graph of P with maximal cardinality is the set of subgraphs induced by the k-faces of P, and, on the other hand, there is a strong dual relation between the cardinality of k-systems and the function $\mathcal{H}_k(\mathcal{O})$. From this relationship polynomially sized proofs (certificates) for the fact that a set of vertex sets indeed is the set of vertex sets of the k-faces are readily obtained. Note that these certificates are purely combinatorial. In particular, no coordinates are involved.

Furthermore, we prove that every acyclic orientation which induces a unique sink in every 2-face of P is an AOF-orientation (Theorem 5). This reveals a strong relationship between the 2-faces and the abstract objective functions of a simple polytope; they can be exploited as certificates for each other. The special role which is played by the 2-skeleton reflects the well-known fact that it is straightforward to reconstruct a simple polytope from its 2-skeleton.

We refer to Ziegler's book [10] for a detailed treatment of all notions and concepts we rely on.

2. Results

Let P be a simple d-polytope. We denote the graph of P by $\mathcal{G}_P = (\mathcal{V}(P), \mathcal{E}(P))$, where $\mathcal{V}(P)$ is the set of vertices of P and $\mathcal{E}(P)$ is the set of its edges.

If $W \subseteq \mathcal{V}(P)$ is a subset of vertices, then $\mathcal{G}_P(W)$ is the subgraph of $\mathcal{G}_P$ induced by W. For each k let $\mathcal{V}_k(P)$ be the set of vertex sets of k-faces of P. As usual, $f_k(P) := |\mathcal{V}_k(P)|$ is the number of k-faces of P. We will often identify a face F of P with the subgraph of $\mathcal{G}_P$ (denoted by $\mathcal{G}_P(F)$) that is induced by the vertices of F.

Definition 1: Let P be a simple d-polytope and $0 \le k \le d$.

(i) A **k-frame** of P is a (not necessarily induced) subgraph of $\mathcal{G}_P$ isomorphic

to the star $K_{1,k}$, where the vertex of degree $k \geq 2$ is called the **root** of the k-frame.

(ii) A set $\mathcal{S}$ of subsets of $\mathcal{V}(P)$ is called a **k-system** of $\mathcal{G}_P$ if for every set $S \in \mathcal{S}$ the subgraph $\mathcal{G}_P(S)$ of $\mathcal{G}_P$ is k-regular and the node set of every k-frame of P is contained in a unique set from $\mathcal{S}$.

Obviously, $\mathcal{V}_k(P)$ is a k-system. In general, $\mathcal{V}_k(P)$ is not the only k-system of $\mathcal{G}_P$. Figure 1 shows a 2-system of the graph of a simple 3-polytope P that is different from $\mathcal{V}_2(P)$. We will characterize $\mathcal{V}_k(P)$ among the k-systems of $\mathcal{G}_P$ by means of certain acyclic orientations.
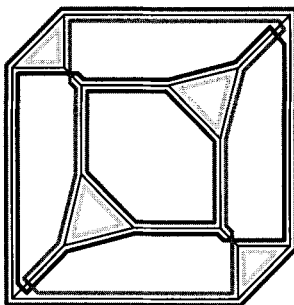

Figure 1. A 2-system (indicated by the subgraphs induced by its sets) that is not the set of vertex sets of 2-faces. The polytope arises from cutting off half of the vertices of the 3-cube.

Let $\mathcal{O}$ be an acyclic orientation of $\mathcal{G}_P$. It is an elementary (but crucial) fact that for every $W \subseteq \mathcal{V}(P)$ the orientation of $\mathcal{G}_P(W)$ induced by $\mathcal{O}$ has at least one sink; furthermore, from each $w \in W$ there is a directed path in $\mathcal{G}_P(W)$ to one of these sinks. Let $h_i(\mathcal{O})$ be the number of vertices of $\mathcal{G}_P$ with precisely i its incident edges directed inwards. We define (for all $0 \leq k \leq d$)

$$\mathcal{H}_k(\mathcal{O}) := \sum_{i=0}^{d} h_i(\mathcal{O}) \binom{i}{k} \quad \text{and} \quad \mathcal{H}(\mathcal{O}) := \sum_{i=0}^{d} h_i(\mathcal{O}) 2^i = \sum_{k=0}^{d} \mathcal{H}_k(\mathcal{O}).$$

The sum $\mathcal{H}_k(\mathcal{O})$ is the number of k-frames for which all edges are directed towards the root. Thus $\mathcal{H}_k(\mathcal{O})$ is the total number of sinks induced in the subgraphs $\mathcal{G}_P(S)$ of $\mathcal{G}_P$ ($S \in \mathcal{S}$) for any k-system $\mathcal{S}$.

One of the beautiful steps on Kalai's "Simple Way to Tell a Simple Polytope from its Graph" [8] (see also [10], Chap. 3.4) is the observation that the AOF-orientations of $\mathcal{G}_P$ are precisely those orientations that minimize $\mathcal{H}(\mathcal{O})$. Theorem 5 below implies that AOF-orientations can also be characterized as those

acyclic orientations of $\mathcal{G}_P$ that minimize $\mathcal{H}_2(\mathcal{O})$. If $\mathcal{O}$ is an AOF-orientation of $\mathcal{G}_P$, then $(h_0(\mathcal{O}), \ldots, h_d(\mathcal{O}))$ is the h-vector of P (see, e.g., [10], Chap. 8.3); in particular, the numbers $h_k(\mathcal{O})$ do not depend on the specific choice of the AOF-orientation $\mathcal{O}$.

There is an important relationship between the k-systems and the acyclic orientations of $\mathcal{G}_P$.

THEOREM 1: *Let P be a simple d-polytope, and let $1 \leq k \leq d$. For every k-system $\mathcal{S}$ of $\mathcal{G}_P$ and every acyclic orientation $\mathcal{O}$ of $\mathcal{G}_P$ the inequalities*

$$|\mathcal{S}| \overset{(1)}{\leq} f_k(P) \overset{(2)}{\leq} \mathcal{H}_k(\mathcal{O})$$

hold, where (1) holds with equality if and only if $\mathcal{S} = \mathcal{V}_k(P)$, and (2) holds with equality if and only if $\mathcal{O}$ induces precisely one sink on every k-face of P.

Proof: Let $\mathcal{S}$ be a k-system of $\mathcal{G}_P$, and let $\mathcal{O}$ be an acyclic orientation of $\mathcal{G}_P$. Since $\mathcal{O}$ is acyclic, $\mathcal{O}$ induces at least one sink in every $S \in \mathcal{S}$. In particular, $\mathcal{H}_k(O) \geq |\mathcal{S}|$ holds. Hence, inequality (2) (together with the characterization of equality) follows with $\mathcal{S} := \mathcal{V}_k(P)$, and inequality (1) is obtained by choosing $\mathcal{O}$ as any AOF-orientation of $\mathcal{G}_P$. It remains to show that $|\mathcal{S}| = f_k(P)$ implies $\mathcal{S} = \mathcal{V}_k(P)$.

Let $\mathcal{S}$ be a k-system of $\mathcal{G}_P$ with $|\mathcal{S}| = f_k(P)$. In order to show $\mathcal{S} = \mathcal{V}_k(P)$ it suffices to prove $\mathcal{V}_k(P) \subseteq \mathcal{S}$. The main ideas of the following are imported from Kalai's paper [8]. Let $W \in \mathcal{V}_k(P)$ be the vertex set of any k-face F of P. There is a linear function (in general position) on P which assigns larger values to the vertices on F than to all other vertices of P. This linear function induces an AOF-orientation $\mathcal{O}$ of $\mathcal{G}_P$ with the property that no edge is directed into W (W is initial).

See Fig. 2 for a sketch of the situation. Denote by $t \in W$ the unique sink induced by $\mathcal{O}$ in $\mathcal{G}_P(W)$, and let $w_1, \ldots, w_k \in W$ be the neighbors of t in F. Let S be the (unique) set in $\mathcal{S}$ containing the k-frame with node set $\{t, w_1, \ldots, w_k\}$. Due to $|\mathcal{S}| = f_k(P) = \mathcal{H}_k(\mathcal{O})$ the orientation $\mathcal{O}$ induces a unique sink in $\mathcal{G}_P(S)$, which must be t.

Since W is initial, this implies $S \subseteq W$ (because there must be a directed path from every vertex in S to t). Hence, $\mathcal{G}_P(S)$ is a k-regular subgraph of the k-regular and connected graph $\mathcal{G}_P(W)$. Thus, $W = S \in \mathcal{S}$.   ∎

As a consequence of Theorem 1, we obtain a characterization of $\mathcal{V}_k(P)$ that is quite similar to Kalai's characterization of AOF-orientations via minimizers of $\mathcal{H}(\mathcal{O})$.
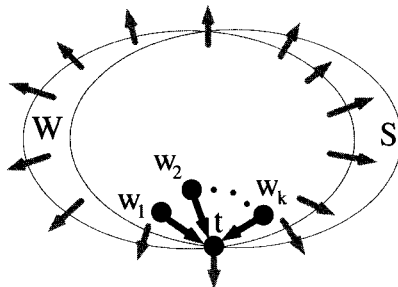
Figure 2. Illustration of the proof of Theorem 1.

COROLLARY 2: *Let P be a simple d-polytope. A k-system of $\mathcal{G}_P$ is $\mathcal{V}_k(P)$ if and only if it has maximal cardinality among all k-systems of $\mathcal{G}_P$.*

Similar to Kalai's result, this corollary implies that the set $\mathcal{V}_{d-1}(P)$ of vertex sets of facets of P can be computed from $\mathcal{G}_P$. However, it does not shed any light on the question, how fast this can be done. From the complexity point of view, the next characterization (which follows from Theorem 1, since every simple d-polytope has an AOF-orientation) is much more valuable.

COROLLARY 3: *Let P be a simple d-polytope, and let $\mathcal{S}$ be a k-system of $\mathcal{G}_P$ (with $k \leq 1$). Then either there is an acyclic orientation $\mathcal{O}$ of $\mathcal{G}_P$ with $\mathcal{H}_k(\mathcal{O}) = |\mathcal{S}|$ or there is a k-system $\mathcal{S}'$ of $\mathcal{G}_P$ with $|\mathcal{S}'| > |\mathcal{S}|$. In the first case, $\mathcal{S} = \mathcal{V}_k(P)$, in the second, $\mathcal{S} \neq \mathcal{V}_k(P)$.*

Corollary 3 yields a good characterization of $\mathcal{V}_k(P)$ among all sets of subsets of vertices of a simple polytope P (given by its graph) in the sense of Edmonds [4, 5]: for every subset $\mathcal{S}$ of vertex sets of P one can efficiently prove the answer to the question "Is $\mathcal{S} = \mathcal{V}_k(P)$?" (although it is currently unknown if one can also find the answer efficiently). If the answer is "yes," then we may prove this in polynomially many steps (in the sizes of $\mathcal{G}_P$ and $\mathcal{S}$) by first checking that $\mathcal{S}$ is a k-system, and then exhibiting an acyclic orientation $\mathcal{O}$ of $\mathcal{G}_P$ with $|\mathcal{S}| = \mathcal{H}_k(\mathcal{O})$. If the answer is "no," then we may prove this by showing that $\mathcal{S}$ is not a k-system of $\mathcal{G}_P$, or, if it is a k-system, by exhibiting a larger k-system $\mathcal{S}'$ of $\mathcal{G}_P$.

Since the number of facets of a simple d-polytope P is bounded by a polynomial in the size of $\mathcal{G}_P$, Corollary 3 also implies that the question whether a given single subset of vertices is the vertex set of some facet of P has a good characterization.

It had been hoped for a long time that such good characterizations (for $k = d - 1$) would be obtained by an eventual proof of a conjecture due to Perles. Already in 1970 he conjectured that every subset of vertices of a simple d-polytope P which induces a $(d-1)$-regular, connected, non-separating subgraph of $\mathcal{G}_P$ is the vertex set of a facet of P. This would even imply much more than good characterizations: it would immediately yield polynomial time algorithms to decide whether a set of vertices is the vertex set of a facet, and whether a subset of sets of vertices is $\mathcal{V}_{d-1}(P)$. However, recently Haase and Ziegler [6] disproved Perles' conjecture.

For $k = 2$, Theorem 1 (together with Theorem 5) also provides us with a good characterization of the AOF-orientations among all acyclic orientations of the graph $\mathcal{G}_P$ of a simple polytope P (see Corollary 4). Previously, the only method that was known to prove that an acyclic orientation of $\mathcal{G}_P$ is an AOF-orientation was to show that it minimizes $\mathcal{H}(\mathcal{O})$ by exploring all acyclic orientations of $\mathcal{G}_P$ (where it was perhaps the most striking result of [8] that such a method does exist at all). Notice that, if in addition to $\mathcal{G}_P$, also $\mathcal{V}_{d-1}(P)$ is specified as input, then it can be decided in polynomial time whether an acyclic orientation of $\mathcal{G}_P$ is an AOF-orientation. This follows easily from the equivalence between AOF's on P and shellings of P^Δ. However, in our context the polytope P is specified just by its graph, and the ultimate question is whether $\mathcal{V}_{d-1}(P)$ can be computed efficiently at all.

COROLLARY 4: *Let P be a simple d-polytope, and let $\mathcal{O}$ be an acyclic orientation of $\mathcal{G}_P$. Then either there is a 2-system $\mathcal{S}$ of $\mathcal{G}_P$ with $|\mathcal{S}| = \mathcal{H}_2(\mathcal{O})$ or there is an acyclic orientation $\mathcal{O}'$ of $\mathcal{G}_P$ with $\mathcal{H}_2(\mathcal{O}') < \mathcal{H}_2(\mathcal{O})$. In the first case, $\mathcal{O}$ is an AOF-orientation, in the second, it is not.*

While the "either or" statement of the corollary follows immediately from Theorem 1, the fact that in the first case $\mathcal{O}$ is an AOF-orientation is implied by the following result (which, in particular, implies that Ex. 8.12 (iv) in [10] cannot be solved). The theorem had already been proved for hypercubes by [7]. For 3-dimensional simple polytopes the result of Theorem 5 has been obtained independently by Develin [3].

THEOREM 5: *Let P be a simple polytope, and let $\mathcal{O}$ be an acyclic orientation of $\mathcal{G}_P$. If $\mathcal{O}$ induces precisely one sink on every 2-face of P, then $\mathcal{O}$ is an AOF-orientation.*

Since every face of a simple polytope is simple, Theorem 5 follows immediately from the following result.

LEMMA 6: *Let P be a simple d-polytope and $k \geq 2$. If $\mathcal{O}$ is an acyclic orientation of $\mathcal{G}_P$ that has more than one global sink, then there is a k-face of P on which $\mathcal{O}$ induces more than one sink.*

Proof: Let $\mathcal{O}$ have more than one sink in $\mathcal{G}_P$. We denote by $A \subseteq \mathcal{V}(P)$ the set of all vertices from which two different sinks can be reached on directed paths. Since $\mathcal{G}_P$ is connected, $A \neq \emptyset$. Thus we can choose a vertex $a \in A$ which is a sink in $\mathcal{G}_P(A)$, together with two directed paths $(a, b_1, \ldots, t_1)$ and $(a, b_2, \ldots, t_2)$ connecting a with two distinct global sinks t_1 and t_2 (see Fig. 3 for an illustration of the proof).
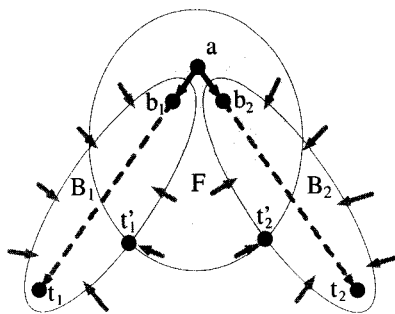

Figure 3. Illustration of the proof of Lemma 6.

Since P is simple and $k \geq 2$, there is a k-face F containing a, b_1, and b_2. For $i \in \{1, 2\}$ denote by B_i the set of vertices of $\mathcal{G}_P$ that lie on some directed path from b_i to t_i. The choice of a as a sink in $\mathcal{G}_P(A)$ implies $B_1 \cap B_2 = \emptyset$. Since both $B_1 \cap F$ and $B_2 \cap F$ are non-empty, the acyclic orientation $\mathcal{O}$ thus induces two distinct sinks t_1' and t_2' in $\mathcal{G}_P(B_1 \cap F)$ and $\mathcal{G}_P(B_2 \cap F)$, respectively. Again, since a is a sink in $\mathcal{G}_P(A)$, both B_1 and B_2 are terminal (no edges are directed outwards). Hence t_1' and t_2' are two distinct sinks in $\mathcal{G}_P(F)$ as well. ∎

It is not too hard to find examples showing that there is no analogue of Theorem 5 for k-faces with $k > 2$. Theorem 5 thus shows that the 2-faces of a simple polytope P play a distinguished role with respect to the AOF-orientations of $\mathcal{G}_P$. It is worth noticing that computing $\mathcal{V}_2(P)$ from $\mathcal{G}_P$ is polynomial time equivalent to computing $\mathcal{V}_{d-1}(P)$ from $\mathcal{G}_P$. To see this, it suffices to observe that the obvious bijections between the neighbors of u and the neighbors of v (for every edge $\{u, v\} \in \mathcal{E}(P)$) defined by the 2-faces and the facets of P, respectively, coincide. Thus, instead of considering the problem of computing $\mathcal{V}_{d-1}(P)$ from $\mathcal{G}_P$ one may rather consider the problem of computing $\mathcal{V}_2(P)$ from $\mathcal{G}_P$. The

2-faces are polygons and thus have a simpler structure than the facets, in general. Moreover, they also bear strong connections to the AOF-orientations as stated in Theorem 5.

3. Discussion

A good characterization, as provided by Corollary 3, often indicates that the corresponding (decision) problem $\mathcal{D}$ (given the graph $\mathcal{G}_P$ of a simple polytope P and a k-system $\mathcal{S}$ of $\mathcal{G}_P$; is $\mathcal{S} = \mathcal{V}_k(P)$?) can be solved in polynomial time. In fact, there are many examples of combinatorial optimization problems for which such a good characterization has guided the algorithm design (primal-dual algorithms). In the theory of computational complexity, this corresponds to the fact that for most problems which are known to be contained in the complexity class $\mathcal{NP} \cap \text{co-}\mathcal{NP}$ it is even known that they belong to the class $\mathcal{P}$ of problems solvable in polynomial time (the most prominent exception is the problem of deciding whether an integer number is a prime).

Unfortunately, Corollary 3 does not imply that problem $\mathcal{D}$ is contained in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$, since it is unknown if one can prove resp. disprove efficiently that a given graph is (isomorphic to) the graph of some simple d-polytope. This question is closely related to the Steinitz problem, the problem to decide whether a given lattice is (isomorphic to) the face-lattice of some polytope (with real-algebraic coordinates). The Steinitz problem is known to be $\mathcal{NP}$-hard even in dimension four (Theorem 9.1.2 in [9]). Furthermore, again already in dimension four, there is no polynomial certificate for the Steinitz problem by specifying rational coordinates (Theorem 9.3.3 in [9]). This can be interpreted as indications for the non-existence of a good characterization for the "integrity" of the input data of problem $\mathcal{D}$. Thus, the good characterization of Corollary 3 seems to have no direct complexity theoretical implications. Nevertheless, it might be encouraging or even be exploited for the design of a polynomial time algorithm for problem $\mathcal{D}$.

Theorem 1 shows that for solving problem $\mathcal{D}$ in polynomial time it would suffice to design a polynomial time method for computing $f_k(P)$ from $\mathcal{G}_P$. One way to achieve this could be a polynomial time method for finding any AOF-orientation of $\mathcal{G}_P$. However, it is not even known whether there is a polynomial time algorithm for finding an AOF of a simple d-polytope P given by its entire face-lattice (not even for $d = 4$). Equivalently, there is no polynomial time algorithm known that finds a shelling of an abstract simplicial complex of which one knows that it is isomorphic to the boundary complex of a simplicial polytope.

Thus, an interesting question is the one for alternative ways to calculate $f_k(P)$ from $\mathcal{G}_P$. For instance, it might be easier to find a polynomial algorithm that finds an acyclic orientation of $\mathcal{G}_P$ which has only one sink per k-face, from which one would obtain $f_k(P)$ as well.

These considerations concern the problem of deciding whether a set of candidates actually is the set $\mathcal{V}_{d-1}(P)$ of vertex sets of facets (or, more generally, of k-faces) of a simple polytope P specified by its graph $\mathcal{G}_P$. The genuine question, however, is whether there is a polynomial time algorithm for finding $\mathcal{V}_{d-1}(P)$ from $\mathcal{G}_P$. Corollary 2 shows that one can phrase this problem as a maximization problem. Hence, it might well be that concepts and tools from Combinatorial Optimization (such as the primal-dual method mentioned above) can help to eventually find a "fast way to tell a simple polytope from its graph."

References

[1] H. Achatz and P. Kleinschmidt, *Reconstructing a simple polytope from its graph*, in *Polytopes — Combinatorics and Computation* (G. Kalai and G. M. Ziegler, eds.), DMV Seminar Band, Vol. 29, Birkhäuser, Basel, 2000, pp. 155–165.

[2] R. Blind and P. Mani-Levitska, *On puzzles and polytope isomorphisms*, Aequationes Mathematicae **34** (1987), 287–297.

[3] M. Develin, e-mail conversation, Nov 2000, develin@bantha.org.

[4] J. Edmonds, *Maximum matching and a polyhedron with 0,1-vertices*, Journal of Research of the National Bureau of Standards – B (Mathematics and Mathematical Physics) **69B** (1965), 125–130.

[5] J. Edmonds, *Paths, trees, and flowers*, Canadian Journal of Mathematics **17** (1965), 449–467.

[6] Ch. Haase and G. M. Ziegler, *Examples and counterexamples for Perles' conjecture*, to appear in Discrete Computational Geometry.

[7] P. L. Hammer, B. Simeone, T. M. Liebling and D. de Werra, *From linear separability to unimodality: A hierarchy of pseudo-boolean functions*, SIAM Journal on Discrete Mathematics **1** (1988), 174–184.

[8] G. Kalai, *A simple way to tell a simple polytope from its graph*, Journal of Combinatorial Theory, Series A **49** (1988), 381–383.

[9] J. Richter-Gebert, *Realization Spaces of Polytopes*, Lecture Notes in Mathematics, Vol. 1643, Springer, Berlin, 1996.

[10] G. M. Ziegler, *Lectures on Polytopes*, Springer-Verlag, New York, 1995; revised edition 1998.